

# Manifest $SO(\mathcal{N})$ invariance and $S$ -matrices of three-dimensional $\mathcal{N} = 2, 4, 8$ SYM

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## Abstract

An on-shell formalism for the computation of  $S$ -matrices of SYM theories in three spacetime dimensions is presented. The framework is a generalization of the spinor-helicity formalism in four dimensions. The formalism is applied to establish the manifest  $SO(\mathcal{N})$  covariance of the on-shell superalgebra relevant to  $\mathcal{N} = 2, 4$  and 8 SYM theories in  $d = 3$ . The results are then used to argue for the  $SO(\mathcal{N})$  invariance of the  $S$ -matrices of these theories: a claim which is proved explicitly for the four-particle scattering amplitudes. Recursion relations relating tree amplitudes of three-dimensional SYM theories are shown to follow from their four-dimensional counterparts. The results for the four-particle amplitudes are verified by tree-level perturbative computations and a unitarity based construction of the integrand corresponding to the leading perturbative correction is also presented for the  $\mathcal{N} = 8$  theory. For  $\mathcal{N} = 8$  SYM, the manifest  $SO(8)$  symmetry is used to develop a map between the color-ordered amplitudes of the SYM and superconformal Chern-Simons theories, providing a direct connection between on-shell observables of D2 and M2-brane theories.

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## 1 Introduction and summary

Recent developments have uncovered a wealth of algebraic structures and dualities pertaining to the scattering matrices of supersymmetric Yang-Mills (SYM) theories in  $3 + 1$  dimensions. In the case of  $\mathcal{N} = 4$  SYM in  $d = 4$ , the possibility that the particularly elegant Parke-Taylor form of the maximal helicity violating (MHV) amplitudes [1] is suggestive of deeper symmetries in the theory had been appreciated for a long time [2, 3]. It is only in recent times that it has become clear that the holomorphic form of the MHV amplitudes is a particular manifestation of Yangian, dual-conformal and dual-superconformal symmetries of the  $S$ -matrix of the gauge theory [4, 5, 6, 7, 8] - symmetries which are not manifest at the level of the Lagrangian of the theory (for recent reviews on the subject see [9, 10, 11, 12, 13]). These symmetries also manifest themselves in the form of recursion relations that relate amplitudes with a given number of external legs to products of amplitudes with lower number of external legs at a fixed order in perturbation theory [14]. Given that the usual Feynman diagrammatic

techniques involve thousands of Feynman diagrams even for  $\sim 10$  legs at tree-level, the algebraic relations are a great boon even from a strictly pragmatic computational point of view. Furthermore, as far as loop corrections are concerned, the method of generalized unitarity allows one to compute higher order corrections to a given amplitude using only the minimal on-shell information, providing a much sought-after alternative to the exponential computational complexity encountered by the standard Feynman diagrammatic approach to loop corrections [15, 16, 17]. All these structures, insights and symmetries have contributed to revealing the underlying analytic and algebraic structures of the  $S$ -matrix of the four-dimensional superconformal theory, along with compelling proposals for its form to all orders in perturbation theory [8, 18].

Given this enormous progress in our understanding of the  $S$ -matrix of  $\mathcal{N} = 4$  SYM it is natural to ask if  $S$ -matrices of non-conformal gauge theories continue to exhibit “hidden” structures not captured by their Lagrangians as well. Of particular interest in this regard are SYM theories in  $d = 2 + 1$ . In three dimensions, theories with  $\mathcal{N} = 2, 4$ , and 8 supersymmetry inherit all the Poincaré supersymmetries of the four-dimensional theories of which they are dimensional reductions. However, the appearance of a dimensionful coupling constant prevents them from being conformally invariant, even classically. These theories thus provide a controlled departure from the regime of classically conformally invariant four-dimensional Yang-Mills theories whose scattering matrices have been explored in the greatest detail following the developments alluded to above<sup>1</sup>. Of special interest is the D2-brane worldvolume theory described by the dimensional reduction of  $\mathcal{N} = 4$  SYM to the three-dimensional sixteen-supercharge non-conformal  $\mathcal{N} = 8$  SYM theory. The duality between D2 and M2-brane theories imply that the  $g_{YM}^2 \rightarrow \infty$  limit of this theory is described by a superconformal Chern-Simons theory [21, 22, 23, 24, 25] whose scattering matrix has been shown to exhibit Yangian symmetries and other twistorial properties that are very reminiscent of the  $S$ -matrix of  $\mathcal{N} = 4$  SYM [26, 27, 28, 29]. Thus, it might be expected that the  $S$ -matrix of the three-dimensional gauge theory might contain special structures that its Lagrangian obscures.

On the face of it, a direct connection between the  $S$ -matrices of the two  $\mathcal{N} = 8$  three-dimensional gauge theories presents a challenge. In the case of the gauge group being  $SU(2)$ , the M2-brane theory has a manifest  $SO(8)$  R-symmetry, which is reflected in its  $S$ -matrix. More generally, it may be expected - and it was indeed verified to all loop orders in the case of four-particle amplitudes in [26] - that the scattering matrices of  $\mathcal{N} \geq 4$  SCS theories reflect the global R-Symmetries of their Lagrangians. SYM theories with  $\mathcal{N} = 2, 4$  or 8 supersymmetries, only possess  $SO(\mathcal{N} - 1)$  global R-symmetry in their Lagrangians. Thus, for there to be any meaningful comparison of the D2 and M2-brane scattering matrices it is imperative that we understand how the extra  $U(1)$  symmetry emerges in the Yang-Mills theories. The situation clearly requires the development of on-shell techniques for SYM theories that parallel the recent studies with M2 brane theories.

If one is to look at techniques used to study on-shell properties of four-dimensional gauge theories as models for developing  $d = 2 + 1$  on-shell methods, several ostensible arguments can be made to suggest that four-dimensional SYM techniques do not readily adapt to a three-dimensional context. The fundamental building block of

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<sup>1</sup>For recent progress in various aspects of highly supersymmetric three-dimensional Yang-Mills theories see [19, 20]

$d = 4$  on-shell techniques is the spinor-helicity basis for the vector and spinor degrees of freedom, which is at the heart of many twistorial aspects of the  $S$ -matrices of  $d = 4$  SYM theories. Thus, the absence of a helicity degree of freedom in three dimensions appears to present an immediate stumbling block. Furthermore, the absence of conformal symmetries in the SYM theories of interest to this work appears to rule out dual-conformal and dual-superconformal symmetries – symmetries that were a sufficient condition for the existence of infinite dimensional Yangian symmetries for the  $S$ -matrix of  $\mathcal{N} = 4$  SYM in  $d = 4$  [6] – in  $d=3$ .

Given the motivations and technical caveats discussed above, we develop a manifestly three-dimensional on-shell formalism for SYM theories in this paper. Motivated by the four-dimensional spinor-helicity framework, we use solutions of the massless Dirac equation to construct gluon polarization vectors. One of the advantages of our formalism is that it allows us to explicitly track the fate of the helicity degree of freedom when four-dimensional gauge theories are reduced to  $d = 2 + 1$ . We find that helicity is augmented to a continuous  $U(1)$  symmetry of the  $S$ -matrices of three-dimensional SYM theories. Furthermore, this  $U(1)$  degree of freedom couples to the  $SO(\mathcal{N} - 1)$  R-symmetry of  $\mathcal{N} = 2, 4$  and  $8$   $d = 2 + 1$  SYM theories to make the on-shell representation of the supersymmetry algebra *manifestly*  $SO(\mathcal{N})$  covariant. Put differently, our formalism makes it transparent that the scalar corresponding to the on-shell gluon in three dimensions couples to the remaining  $\mathcal{N} - 1$  scalars of the gauge theories in a way that makes the on-shell supersymmetry algebra the same as the (off-shell) algebra for a free theory of  $\mathcal{N}$  massless real scalars and fermions. This symmetry enhancement has not been known to be manifest in the Lagrangian except in the abelian limit where the gauge field can be dualized into a scalar [30]. What we exhibit in the paper is the same phenomenon; but in the context of the  $S$ -matrices of the corresponding non-abelian gauge theories.

Using our formalism we are able to show that the four-particle amplitudes of  $\mathcal{N} = 2, 4, 8$   $d = 2 + 1$  SYM theories have a manifest  $SO(\mathcal{N})$  invariance to all orders in perturbation theory. We also argue, though we do not provide a formal proof in this paper, that the manifest  $SO(\mathcal{N})$  invariance should extend to the higher-particle amplitudes as well. The four-particle amplitudes have the form<sup>2</sup>

$$S_{YM}(t, s, g_{YM}^2) S_{ijkl}(\{\mathcal{W}\}; t, s). \quad (1.1)$$

where the “universal” term  $S_{ijkl}(\{\mathcal{W}\}; t, s)$  only depends on the species of particles being scattered  $\{\mathcal{W}\}$  and contains all the  $SO(\mathcal{N})$  dependence (in the indices  $i, j, k$  and  $l$ ). We show that  $S_{ijkl}(\{\mathcal{W}\}; t, s)$  is an  $SO(\mathcal{N})$  invariant tensor for all the theories under consideration in this paper. Furthermore, for the special case of the  $\mathcal{N} = 8$  theory with gauge group  $SU(2)$ , we show that  $S_{ijkl}(\{\mathcal{W}\}; t, s)$  is the same as the corresponding quantity for the BLG theory computed in [26]. This analysis allows us to see a direct connection between an  $SO(8)$  invariant physical observable in the D2 and M2-brane theories and reduce the problem of establishing the duality (at least in the context of four-particle amplitudes) to the asymptotic behavior of a single function  $S_{YM}(t, s, g_{YM}^2)$  as  $g_{YM}^2 \rightarrow \infty$ .

With the general aim of studying  $S$ -matrix elements of various three-dimensional theories in perturbation theory we also provide a one-to-one map between  $S$ -matrices of

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<sup>2</sup> $s$  and  $t$  are the standard Mandelstam variables, while  $g_{YM}$  is the coupling constant of the Yang-Mills theory under consideration.

three-dimensional gauge theories and the corresponding matrices of the four-dimensional theories that they are dimensional reductions of. In particular we give a useful presentation of three and four-dimensional gamma matrices and spinor-helicity bases in which all tree-level amplitudes in three-dimensional SYM theories are obtained simply by setting the fourth component of the momentum to zero in the known results for four-dimensional amplitudes. We use the relations between three and four-dimensional amplitudes to derive the analogs of the BCFW relations for three-dimensional SYM theories. We also verify these results by explicit perturbative calculations for the  $d = 2 + 1$  theories of interest. In this context, we are also able to interpret the known kinematical results in four dimensions such as the vanishing of all helicity “plus” amplitudes as BPS conditions in three dimensions. Finally, using the  $\mathcal{N} = 8$  theory as an illustrative example, we are able to show that the integrands corresponding to loop corrections for amplitudes in the three-dimensional theories (obtained using generalized unitarity) are also obtained from known integrands for the associated four-dimensional theories.

The paper is organized as follows. In section 2 we discuss the on-shell SUSY algebra, concentrating on the  $\mathcal{N} = 2$  theory. We show that the algebra, and therefore the  $S$ -matrix, manifests  $SO(2)$  symmetry. The argument readily generalizes to  $SO(\mathcal{N})$  symmetry for theories with  $\mathcal{N} > 2$ . We also give an explicit map of the  $d = 4$  spinor-helicity formalism under dimensional reduction to  $d = 3$  and obtain recursion relations for all tree-level amplitudes. We derive specific relations between four-particle scattering amplitudes following from the SUSY algebra. In section 3 we calculate all four-particle scattering amplitudes in the  $\mathcal{N} = 2, 4, 8$  theories at tree-level directly using Feynman diagrams. The manifest  $SO(\mathcal{N})$  forms of the amplitudes are presented, which allows a verification of the relations derived from the SUSY algebra in section 2. In section 4 we discuss the four-particle amplitudes in the  $\mathcal{N} = 8$  theory at one-loop, recovering the scalar box integral found in  $\mathcal{N} = 4$  SYM in  $d = 4$ . In section 5 we discuss the relationship between the  $S$ -matrix of superconformal Chern-Simons theories and the SYM theories considered in the paper. We conclude the paper with a discussion in section 6.

## 2 On-shell $\mathcal{N} = 2$ algebra in three dimensions

In this section we obtain the  $SO(\mathcal{N})$  symmetric realization of the on-shell supersymmetry algebra and introduce the spinor-“helicity” techniques tailored for the analyses of  $S$ -matrices of  $d = 3$  gauge theories. We then use our particular realization of the algebra to constrain four-particle amplitudes to a single function of the Mandelstam variables and  $g_{YM}^2$ , thereby fixing the “matrix” or R-symmetry structure of the four particle  $S$ -matrix. The relationships so obtained between the various matrix elements are later verified explicitly at weak coupling. We also present an explicit map between the three-dimensional formalism developed in this paper and the well known spinor-helicity framework in four dimensions. As a byproduct of this map, we are able to relate *all* tree-level three-dimensional amplitudes for  $\mathcal{N} = 2, 4$  and 8 theories to known results for  $\mathcal{N} = 1, 2$  and 4 theories in one higher dimension very transparently. In the interests of brevity, most of the discussion concerning our formalism will be limited to the  $d = 3$ ,  $\mathcal{N} = 2$  case. The generalizations of the results presented in this section to

higher supersymmetry are straightforward and we shall present the results relevant to higher extended supersymmetry later in the paper.

The on-shell algebra acting on asymptotic scattering states is nothing but the supersymmetry algebra of the free theory. The free  $\mathcal{N} = 2$  SYM theory can be expressed in a manifestly  $SO(2)$  symmetric form by dualizing the gauge field to a scalar  $\partial_\mu \Phi_1 \sim \epsilon_{\mu\nu\rho} F^{\nu\rho}$ . Note that since the free theory is the same as the abelian limit, we can ignore color indices for this discussion. The dualized action

$$S_d = \int d^3x \left( -\frac{1}{2} \partial_\mu \Phi_I \partial^\mu \Phi_I + \frac{i}{2} \bar{\chi}_I \gamma_\mu \partial^\mu \chi_I \right), \quad (2.1)$$

is invariant under the  $\mathcal{N} = 2$  SUSY transformations

$$\begin{aligned} \delta \chi_I &= -\frac{1}{2} (\partial_\mu \Phi_1 \gamma^\mu \epsilon_I + \epsilon_{IJ} \partial_\mu \Phi_2 \gamma^\mu \epsilon_J), \\ \delta \Phi_1 &= \frac{i}{2} \bar{\chi}_I \epsilon_I, \\ \delta \Phi_2 &= \frac{i}{2} \bar{\chi}_I \epsilon_J \epsilon_{IJ}, \end{aligned} \quad (2.2)$$

where  $\epsilon_{12} = -\epsilon_{21} = +1$ . Also, in the above equations,  $\delta \mathcal{W} = [\bar{\epsilon}_I Q_I, \mathcal{W}]$  and we can read-off the algebra

$$[\bar{\beta}_M Q_M, \bar{\epsilon}_N Q_N] = \frac{1}{2} (\bar{\epsilon}_L \gamma^\mu \beta_L) p_\mu, \quad (2.3)$$

or equivalently

$$\{Q_a^\alpha, Q_b^\beta\} = \frac{1}{2} P^{\alpha\beta} \delta_{ab}, \quad \text{where } P^{\alpha\beta} = -(p_\mu \gamma^\mu C^{-1})^{\beta\alpha}. \quad (2.4)$$

In three dimensions, we can always pick a real Majorana representation for the  $\gamma$  matrices  $\gamma^\mu = (i\sigma^2, \sigma^1, \sigma^3)$  with  $C = \gamma^0$ , so that

$$P^{\alpha\beta} = P^{\beta\alpha} = \begin{pmatrix} -p_0 - p_1 & p_2 \\ p_2 & -p_0 + p_1 \end{pmatrix}. \quad (2.5)$$

The solution of the Dirac equation  $\gamma^\mu p_\mu u(p) = 0$  is given by

$$u(p) = \frac{1}{\sqrt{p_0 - p_1}} \begin{pmatrix} p_2 \\ p_1 - p_0 \end{pmatrix}, \quad u^\alpha(p) u^\beta(p) = -P^{\alpha\beta}. \quad (2.6)$$

The on-shell (momentum space) version of the  $\mathcal{N} = 2$  algebra can be expressed as follows (we use  $a$  and  $\lambda$  to denote the momentum-space creation operators for the  $\Phi$  and  $\chi$  fields respectively)

$$\begin{aligned} Q_I^\alpha |a_1\rangle &= \frac{1}{2} u^\alpha |\lambda_I\rangle, \\ Q_I^\alpha |a_2\rangle &= \frac{1}{2} \epsilon_{IJ} u^\alpha |\lambda_J\rangle, \\ Q_J^\alpha |\lambda_I\rangle &= -\frac{1}{2} u^\alpha (\delta_{JI} |a_1\rangle + \epsilon_{JI} |a_2\rangle). \end{aligned} \quad (2.7)$$

This is the manifestly  $SO(2)$  covariant form of the algebra which should be realized on the asymptotic states of the gauge theory.

## 2.1 Explicit realization

To concretely establish that (2.7) is indeed the on-shell representation of the SUSY algebra for  $\mathcal{N} = 2$  SYM in  $d = 2 + 1$ , in this section we will obtain the result via dimensional reduction, starting with  $\mathcal{N} = 1$  SYM in  $d = 4$ , whose action is given in (A.5). We start with a four-dimensional real representation of the  $\Gamma$  matrices

$$\Gamma^M = \left\{ \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}, \begin{pmatrix} \sigma^1 & 0 \\ 0 & -\sigma^1 \end{pmatrix}, \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}, \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \right\}, \quad (2.8)$$

$$\Gamma^5 = i\Gamma^0 \dots \Gamma^3.$$

The four-dimensional Majorana fermion  $\Psi = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$ . The three-dimensional  $\gamma$  matrices are

$$\gamma^\mu = (i\sigma^2, \sigma^1, \sigma^3). \quad (2.9)$$

The dimensional reduction is carried out by compactifying the “3” direction. It is readily seen that the fermion kinetic energy term  $\int \bar{\Psi} \Gamma^M \partial_M \Psi = \int \bar{\lambda}_I \gamma^\mu \partial_\mu \lambda_I$  upon dimensional reduction, leading us to identify  $\lambda_I$  as the three-dimensional fermions. It is understood that the charge conjugation matrix is identified with  $\Gamma^0$  and  $\gamma^0$  in four and three dimensions respectively and the Majorana conditions  $\Psi^\dagger = \Psi^T$  and  $\lambda_I^\dagger = \lambda_I^T$  are imposed. The four-dimensional SUSY transformation law  $\delta A_M = \frac{1}{2} \bar{\epsilon} \Gamma_M \Psi$  (we will not display the color indices in this subsection to avoid notational clutter) translates into the three-dimensional relation

$$\begin{aligned} \delta A_\mu &= \frac{1}{2} \bar{\eta}_I \gamma_\mu \lambda_I, \\ \delta \Phi &= \frac{1}{2} \epsilon_{IJ} \bar{\eta}_I \lambda_J, \end{aligned} \quad (2.10)$$

where  $A_3 = \Phi$ . We now want to translate these relations into relations between momentum space physical degrees of freedom and recover (2.7). For this purpose, it is very convenient to introduce a polarization vector

$$\epsilon_\mu(p, k) = \frac{\langle p | \gamma_\mu | k \rangle}{\langle kp \rangle}, \quad p_\mu \epsilon^\mu(p, k) = k_\mu \epsilon^\mu(p, k) = 0. \quad (2.11)$$

It is implied that

$$|p\rangle = u(p), \quad \langle p| = \bar{u}(p), \quad \langle kp\rangle = \bar{u}(k)u(p) = -\langle pk\rangle, \quad (2.12)$$

where  $u(p)$  is the wavefunction defined before. The polarization vectors satisfy

$$\epsilon_\mu(p, k) \epsilon^\mu(p, k') = +1, \quad \epsilon_\mu(p, k) (\gamma^\mu)_{\alpha\beta} = \frac{2\bar{u}_\beta(p)u_\alpha(k) - \delta_{\alpha\beta}\langle pk\rangle}{\langle kp\rangle}. \quad (2.13)$$

To get the second equation above, we have used the Fierz identity:  $(\gamma_\mu)_{\alpha\beta}(\gamma^\mu)_{\gamma\delta} = 2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta}$ .

The dynamical fields are mode-expanded as follows,

$$\begin{aligned}
\Phi &= \int \frac{d^2 p}{(2\pi)^2} \frac{1}{\sqrt{2p^0}} \left( a_2^\dagger(p) e^{ip \cdot x} + a_2(p) e^{-ip \cdot x} \right), \\
A_\mu &= \int \frac{d^2 p}{(2\pi)^2} \frac{1}{\sqrt{2p^0}} \epsilon_\mu(p, k) \left( a_1^\dagger(p) e^{ip \cdot x} + a_1(p) e^{-ip \cdot x} \right), \\
\lambda_I &= \int \frac{d^2 p}{(2\pi)^2} \frac{1}{\sqrt{2p^0}} \left( u(p) \lambda_I^\dagger(p) e^{ip \cdot x} + u(p) \lambda_I(p) e^{-ip \cdot x} \right).
\end{aligned} \tag{2.14}$$

Application of (2.10) to the oscillator expansion given above implies that the on-shell states  $a_I^\dagger|0\rangle = |a_I\rangle$  transform as

$$\begin{aligned}
Q_I^\alpha |a_1\rangle &= \frac{1}{2} u^\alpha |\lambda_I\rangle, \\
Q_I^\alpha |a_2\rangle &= \frac{1}{2} \epsilon_{IJ} u^\alpha |\lambda_J\rangle,
\end{aligned} \tag{2.15}$$

which are the first two equations of (2.7). The action for the supercharge on the fermion field

$$Q_J^\alpha |\lambda_I\rangle = -\frac{1}{2} u^\alpha (\delta_{IJ} |a_1\rangle + \epsilon_{JI} |a_2\rangle), \tag{2.16}$$

follows simply from the condition of the closure of the algebra (2.4). We have thus recovered the manifestly  $SO(2)$  symmetric form of the on-shell  $\mathcal{N} = 2$  algebra directly from the canonical quantization of the gauge theory. The  $S$ -matrix

$$\langle \mathcal{W}_{I_1} \mathcal{W}_{I_2} \cdots \mathcal{W}_{I_n} \rangle = S_{I_1 I_2 \cdots I_n}, \tag{2.17}$$

where  $\mathcal{W}_J$  stand for any of the four bosonic or fermionic fields must be  $SO(2)$  invariant, since the component fields have a manifest  $SO(2)$  covariance and the  $S$ -matrix commutes with the supercharges given above. We generalize these arguments in section 2.5 to higher extended supersymmetry algebras.

## 2.2 Recovering helicity

Equivalently, one can recast the algebra in terms of a  $U(1)$  symmetric form, which is very instructive for the purposes of drawing parallels with known results for  $\mathcal{N} = 1$  SYM in  $d = 4$ . Defining the complex combinations  $\mathcal{W}_\pm = \frac{1}{\sqrt{2}}(\mathcal{W}_1 \pm i\mathcal{W}_2)$ , we can express the algebra on single particle states as

$$\begin{aligned}
Q_+^\alpha |a_+\rangle &= \frac{1}{\sqrt{2}} u^\alpha |\lambda_+\rangle, & Q_+^\alpha |\lambda_-\rangle &= -\frac{1}{\sqrt{2}} u^\alpha |a_-\rangle, \\
Q_-^\alpha |a_-\rangle &= \frac{1}{\sqrt{2}} u^\alpha |\lambda_-\rangle, & Q_-^\alpha |\lambda_+\rangle &= -\frac{1}{\sqrt{2}} u^\alpha |a_+\rangle, \\
Q_-^\alpha |a_+\rangle &= Q_+^\alpha |a_-\rangle = Q_+^\alpha |\lambda_+\rangle = Q_-^\alpha |\lambda_-\rangle = 0.
\end{aligned} \tag{2.18}$$

The  $a_\pm$  ( $\lambda_\pm$ ) states are direct analogues of the  $d = 4$  gluon (gluino) helicities. In the next subsection we consider the fate of the helicity degree of freedom under dimensional reduction from  $d = 4$  in detail.



### 2.3 Dimensional reduction of the spinor-helicity basis from $d = 4$ to $d = 3$

The on-shell supersymmetry transformations given above were derived in a purely three-dimensional set up. We now study how the above discussion is related to the spinor-helicity framework in four dimensions. With the four-dimensional gamma matrices as in (2.8), eigenstates of the helicity operator  $\Gamma_{\pm} = \frac{1}{2}(1 \pm \Gamma^5)$  are given by spinors of the form

$$\begin{aligned} U_+ &= \frac{1}{\sqrt{2}} \begin{pmatrix} V \\ -iV \end{pmatrix}, \quad \text{where} \quad p_{\mu} \gamma^{\mu} V = +ip_3 V, \\ U_- &= \frac{1}{\sqrt{2}} \begin{pmatrix} U \\ iU \end{pmatrix}, \quad \text{where} \quad p_{\mu} \gamma^{\mu} U = -ip_3 U. \end{aligned} \quad (2.19)$$

The four-dimensional massless Dirac equation, written on the right of the above equations is readily interpreted as two copies of the massive Dirac equation satisfied by  $U$  and  $V$ , with masses  $\sim \pm p_3$ . The solutions for  $U$  and  $V$  are

$$V = \frac{1}{\sqrt{p_0 - p_1}} \begin{pmatrix} p_2 + ip_3 \\ p_1 - p_0 \end{pmatrix}, \quad U = \frac{1}{\sqrt{p_0 - p_1}} \begin{pmatrix} p_2 - ip_3 \\ p_1 - p_0 \end{pmatrix}. \quad (2.20)$$

We also have the closure<sup>3</sup> conditions  $V^{\alpha}(p)U^{\beta}(p) = -P^{\alpha\beta} - ip_3\epsilon^{\alpha\beta}$ . The two wavefunctions are related by complex conjugation as  $U^*(p) = -iV(-p)$ . Most importantly for our considerations  $U(p)_{p_3=0} = V(p)_{p_3=0} = u(p)$ , where the momentum of the last wavefunction is three-dimensional. The four-dimensional spinor products can be expressed as

$$\langle pq \rangle = +\bar{U}(p)V(q), \quad [pq] = +\bar{V}(p)U(q), \quad |\langle pq \rangle|^2 = -2p \cdot q. \quad (2.21)$$

The four-dimensional polarization vectors are chosen in-line with standard conventions

$$\epsilon_M^{\pm}(p, k) = \pm \frac{\langle p \pm | \Gamma_M | k \pm \rangle}{\sqrt{2} \langle k \mp | p \pm \rangle}, \quad (2.22)$$

where  $k$  is a reference four-momentum, and decompose, upon dimensional reduction to

$$\epsilon^{\pm\mu}(p, k) = +\frac{1}{\sqrt{2}}\epsilon^{\mu}(p, k), \quad \epsilon^{+3}(p, k) = -\frac{i}{\sqrt{2}}, \quad \epsilon^{-3}(p, k) = +\frac{i}{\sqrt{2}}. \quad (2.23)$$

The explicit split of the four-dimensional photon into a three-dimensional photon and a scalar upon the compactification of the “3” direction is as follows

$$\begin{aligned} A^M &= \int \frac{d^2 p}{(2\pi)^2} \frac{1}{\sqrt{2p^0}} \epsilon^{\pm M}(p, k) A^{\pm\dagger}(p) e^{ip \cdot x} + \text{h.c.} \\ &= \int \frac{d^2 p}{(2\pi)^2} \frac{1}{2\sqrt{p^0}} [\epsilon^{\pm\mu}(p, k) (A^{+\dagger}(p) + A^{-\dagger}(p)) - i (A^{+\dagger}(p) - A^{-\dagger}(p))] e^{ip \cdot x} + \text{h.c.} \end{aligned} \quad (2.24)$$

---

<sup>3</sup>The solutions  $U, V$  can also be regarded as wavefunctions for massive fermions in three dimensions, with  $p_3 \sim m$ . Indeed, in [26] these solutions were used extensively in the computations of four-particle amplitudes of massive SCS theories.

We can thus readily identify

$$a_1^\dagger = \frac{1}{\sqrt{2}}(A^+ + A^-)^\dagger, \quad a_2^\dagger = -\frac{i}{\sqrt{2}}(A^+ - A^-)^\dagger, \quad a^{\pm\dagger} = (A^\pm)^\dagger. \quad (2.25)$$

Consequently we can relate *all* tree-level  $n$ -particle scalar/gluon amplitudes as

$$\langle ++---+\cdots \rangle|_{d=3} = \langle ++---+\cdots \rangle|_{d=4, x_3=0}. \quad (2.26)$$

Similar formulae exist for amplitudes involving fermions, namely

$$|\lambda_+\rangle = |\Psi_+\rangle_{x_3=0}, \quad |\lambda_-\rangle = |\Psi_-\rangle_{x_3=0}. \quad (2.27)$$

Finally, if the field theory contains scalar degrees of freedom  $\Phi$ , then one trivially has

$$|\Phi\rangle_{d=4, x_3=0} = |\Phi\rangle_{d=3}, \quad (2.28)$$

as the identifying relation between the three-dimensional on-shell state and the dimensional reduction of its four-dimensional counterpart.

These identifications allow us to read-off all the  $\mathcal{N} = 2, 4$  and  $8$   $d = 3$  tree-level amplitudes from the known results for  $\mathcal{N} = 1, 2$  and  $4$  SYM in  $d = 4$ . The spinor bases in three and four dimensions chosen here enable the map between the three and four-dimensional amplitudes to be as simple as possible. Given an amplitude in the four-dimensional theory, one simply sets the fourth component of the momentum to zero to read-off the three-dimensional results.

**Recursion relations:** The map between the three and four-dimensional amplitudes also allows us to readily derive tree-level recursion relations for three-dimensional amplitudes. Using the notation of [14] and given a four-dimensional amplitude  $\mathcal{A}$ , we define the amplitude with the momenta shifted by a complex parameter  $z$ ,  $\mathcal{A}(z)$ . Taking the limit where the fourth components of all the momenta are set to zero, after subjecting the shifted amplitude to the BCFW relations yields recursion relations for the three-dimensional amplitudes. In other words,

$$\lim_{k_i^3 \rightarrow 0} \left( \frac{1}{2\pi i} \oint dz \frac{\mathcal{A}(z)}{z} = \mathcal{A}(0) - \sum_{ij} \sum_h \frac{\mathcal{A}_L^h(z_{ij}) \mathcal{A}_R^{-h}(z_{ij})}{P_{ij}^2} \right), \quad (2.29)$$

is the relevant recursion relation for SYM amplitudes in  $d = 3$ . Thus to find the recursion relations for a given three-dimensional amplitude, we can “oxidize” it to its four-dimensional counterpart using the dictionary given above, apply the BCFW relations and then dimensionally reduce back to three dimensions to obtain the desired recursion relations.

## 2.4 Relations between amplitudes

We can use the complex form of the algebra to relate various amplitudes. We start out by noticing that all amplitudes of the form  $\langle a_+ \cdots a_+ \rangle$ ,  $\langle \lambda_+ \cdots \lambda_+ \rangle$  and their complex conjugates are annihilated by half of the supercharges, making them  $\frac{1}{2}$  BPS states. Furthermore, they are identically zero to all orders in perturbation theory. For example

$$\langle [Q_-, \lambda_+ a_+ \cdots a_+] \rangle = 0 \Rightarrow \langle a_+ \cdots a_+ \rangle = 0. \quad (2.30)$$

These are then the analogues of all helicity “plus” amplitudes in the four-dimensional case. Similarly, acting with  $Q_-$  allows us to see that all boson amplitudes with only one  $a_-$  – the analogues of the minimally helicity violating amplitudes in four dimensions – are also zero. Turning our attention to four-particle amplitudes, it is easy to see that the only non-vanishing amplitudes are those that involve two plus and two minus fields. Furthermore, the only non-vanishing mixed amplitudes are those that involve bosons and fermions of both signs, i.e. of the type  $\langle a_+ a_- \lambda_+ \lambda_- \rangle$ . These amplitudes are related to the four boson amplitudes as follows

$$\begin{aligned}
\langle \lambda_+ \lambda_- a_+ a_- \rangle &= +\frac{\langle 32 \rangle}{\langle 31 \rangle} \langle a_+ a_- a_+ a_- \rangle, & \langle \lambda_+ \lambda_- a_- a_+ \rangle &= +\frac{\langle 42 \rangle}{\langle 41 \rangle} \langle a_+ a_- a_- a_+ \rangle, \\
\langle a_+ \lambda_- \lambda_+ a_- \rangle &= -\frac{\langle 12 \rangle}{\langle 13 \rangle} \langle a_+ a_- a_+ a_- \rangle, & \langle a_+ \lambda_- a_- \lambda_+ \rangle &= -\frac{\langle 12 \rangle}{\langle 14 \rangle} \langle a_+ a_- a_- a_+ \rangle, \\
\langle \lambda_+ a_+ \lambda_- a_- \rangle &= +\frac{\langle 23 \rangle}{\langle 21 \rangle} \langle a_+ a_+ a_- a_- \rangle, & \langle a_+ \lambda_+ \lambda_- a_- \rangle &= +\frac{\langle 13 \rangle}{\langle 12 \rangle} \langle a_+ a_+ a_- a_- \rangle, \\
\langle \lambda_+ a_- \lambda_- a_+ \rangle &= +\frac{\langle 43 \rangle}{\langle 41 \rangle} \langle a_+ a_- a_- a_+ \rangle, & \langle a_+ a_- \lambda_- \lambda_+ \rangle &= -\frac{\langle 13 \rangle}{\langle 14 \rangle} \langle a_+ a_- a_- a_+ \rangle, \\
\langle a_+ a_- \lambda_+ \lambda_- \rangle &= +\frac{\langle 14 \rangle}{\langle 13 \rangle} \langle a_+ a_- a_+ a_- \rangle, & \langle \lambda_+ a_- a_+ \lambda_- \rangle &= +\frac{\langle 34 \rangle}{\langle 31 \rangle} \langle a_+ a_- a_+ a_- \rangle, \\
\langle a_+ \lambda_+ a_- \lambda_- \rangle &= +\frac{\langle 14 \rangle}{\langle 12 \rangle} \langle a_+ a_+ a_- a_- \rangle, & \langle \lambda_+ a_+ a_- \lambda_- \rangle &= -\frac{\langle 31 \rangle}{\langle 34 \rangle} \langle a_+ a_+ a_- a_- \rangle.
\end{aligned} \tag{2.31}$$

The rest of the two-boson two-fermion amplitudes are related to the ones given above by complex conjugation. Proceeding to the four-fermion amplitudes, we find the following relations

$$\begin{aligned}
\langle \lambda_+ \lambda_+ \lambda_- \lambda_- \rangle &= +\frac{\langle 12 \rangle}{\langle 24 \rangle} \langle a_+ \lambda_+ \lambda_- a_- \rangle = +\frac{\langle 13 \rangle}{\langle 24 \rangle} \langle a_+ a_+ a_- a_- \rangle, \\
\langle \lambda_+ \lambda_- \lambda_+ \lambda_- \rangle &= +\frac{\langle 42 \rangle}{\langle 41 \rangle} \langle a_+ a_- \lambda_+ \lambda_- \rangle = +\frac{\langle 24 \rangle}{\langle 13 \rangle} \langle a_+ a_- a_+ a_- \rangle, \\
\langle \lambda_+ \lambda_- \lambda_- \lambda_+ \rangle &= +\frac{\langle 14 \rangle}{\langle 42 \rangle} \langle a_+ a_- \lambda_- \lambda_+ \rangle = +\frac{\langle 13 \rangle}{\langle 24 \rangle} \langle a_+ a_- a_- a_+ \rangle.
\end{aligned} \tag{2.32}$$

These relationships imply that there is only one independent four-particle amplitude for the  $\mathcal{N} = 2$  theory in three dimensions. The number of independent amplitudes stays the same for theories with higher extended supersymmetries as well. All the above mentioned relationships have been verified explicitly at tree-level, see section 3.

## 2.5 Higher extended on-shell supersymmetry algebras

Here we outline how the methods presented above can be used to derive the on-shell supersymmetry algebra for theories with higher extended supersymmetry, using the  $\mathcal{N} = 4$  supersymmetric case as an illustrative example. The  $\mathcal{N} = 8$  case follows similarly; we have relegated the analogous details to the appendices. We start with  $\mathcal{N} = 1$  SYM in  $d = 6$  with the action

$$S = \int d^6 x \left( -\frac{1}{4} F_{MN}^a F^{aMN} + \frac{i}{2} \bar{\Psi}^a \Gamma'_M D^M \Psi^a \right), \tag{2.33}$$

invariant under

$$\begin{aligned}\delta A_M^a &= \frac{i}{2}(\bar{\Psi}^a \Gamma'_M \epsilon - \bar{\epsilon} \Gamma'_M \Psi^a), \\ \delta \Psi^a &= -\frac{1}{4}[\Gamma'_M, \Gamma'_N] F^{aMN} \epsilon.\end{aligned}\tag{2.34}$$

The six-dimensional gamma matrices are related to the four-dimensional ones (2.8) as

$$\Gamma' = \sigma^1 \otimes \Gamma, \quad \sigma^1 \otimes \Gamma^5, \quad \sigma^2 \otimes \mathbb{1}.\tag{2.35}$$

In six dimensions, one can only have a Weyl condition,  $(1 + \Gamma'^0 \dots \Gamma'^5) \Psi = 0$ , which is satisfied by

$$\Psi = \begin{pmatrix} 0 \\ \Lambda \end{pmatrix},\tag{2.36}$$

where  $\Lambda$  is a four-dimensional Dirac fermion, which can be decomposed into two real Majorana fermions as  $\Lambda = \Lambda^1 + i\Lambda^2$ , where  $\Lambda^{iT} = \Lambda^{i\dagger}$ . The Majorana fermions can be further decomposed into four, three-dimensional Majorana fermions  $\lambda_A$  with  $A = 1, \dots, 4$ , as

$$\Lambda^1 = \begin{pmatrix} \lambda_1 \\ \lambda_3 \end{pmatrix}, \quad \Lambda^2 = \begin{pmatrix} \lambda_2 \\ \lambda_4 \end{pmatrix}.\tag{2.37}$$

After dimensionally reducing the theory to three dimensions and using the spinor formalism described above, we can read-off the action of the supersymmetry generators on the on-shell degrees of freedom from (2.34). The result is

$$Q_A^\alpha |a_B\rangle = \frac{1}{2} u^\alpha \rho_{AC}^B |\lambda_C\rangle, \quad Q_A^\alpha |\lambda_B\rangle = -\frac{1}{2} u^\alpha \rho_{AB}^C |a_C\rangle,\tag{2.38}$$

where  $a_{A=1}$  represents the  $d = 3$  gluon and  $a_{A \neq 1}$  the three scalars of the theory, and where<sup>4</sup>

$$\rho_{BC}^A = \left\{ \mathbb{1} \otimes \mathbb{1}, i\sigma^2 \otimes \mathbb{1}, -\sigma^1 \otimes i\sigma^2, \sigma^3 \otimes i\sigma^2 \right\}.\tag{2.39}$$

On any of the bosonic or fermionic states  $|\mathcal{W}\rangle$  the algebra closes in an  $SO(4)$  symmetric form as

$$\{(Q_A)^\alpha, (Q_B)^\beta\} |\mathcal{W}\rangle = +\frac{1}{2} P^{\alpha\beta} \delta_{AB} |\mathcal{W}\rangle.\tag{2.40}$$

In this form the  $SO(4)$  covariance of the  $\mathcal{N} = 4$  supersymmetry algebra is manifest. As a matter of fact, it is readily seen that this algebra is a symmetry of the  $SO(4)$  invariant free  $\mathcal{N} = 4$  action  $S = \int -\frac{1}{2} \partial_\mu \Phi_A \partial^\mu \Phi_A + \frac{i}{2} \bar{\lambda}_A \gamma_\mu \partial^\mu \lambda_A$ . Thus the four-particle scattering matrix  $\langle \mathcal{W}_A \mathcal{W}_B \mathcal{W}_C \mathcal{W}_D \rangle$  must have the form

$$\langle \mathcal{W}_A \mathcal{W}_B \mathcal{W}_C \mathcal{W}_D \rangle = \mathcal{A} \delta_{AB} \delta_{CD} + \mathcal{B} \delta_{AC} \delta_{BD} + \mathcal{C} \delta_{AD} \delta_{BC} + \mathcal{D} \epsilon_{ABCD},\tag{2.41}$$

for it to commute with the supercharges given above. More generally,  $n$ -particle amplitudes of this gauge theory must only involve  $SO(\mathcal{N})$  invariants for the  $S$ -matrix to commute with the on-shell supersymmetry generators. The relations between the undetermined coefficients as well as the extension of the formalism to the case of  $\mathcal{N} = 8$  SUSY is discussed in the next chapter, where the perturbative results for  $\mathcal{N} = 2, 4$ , and 8 theories are presented in a unified manner.

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<sup>4</sup>The upper index on  $\rho_{BC}^A$  labels the elements in the list, so that the lower indices are the indices of the  $4 \times 4$  matrices.

### 3 Tree-level four particle amplitudes

In order to have an explicit check of the relations put forth in the preceding sections, in this section we will give results for four-particle scattering in  $\mathcal{N} = 2, 4$  and  $\mathcal{N} = 8$  SYM at tree-level, from a direct Feynman diagram calculation. Our conventions are collected in appendix A.

The SYM action may be expressed<sup>5</sup> (in mostly positive signature) as

$$S = \frac{1}{g^2} \text{Tr} \int d^3x \left( -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} - D_\mu \Phi_i D^\mu \Phi_i + i \bar{\lambda}_A \gamma^\mu D_\mu \lambda_A + \rho_{AB}^i \bar{\lambda}_A [\Phi_i, \lambda_B] \right), \quad (3.1)$$

where  $\lambda_A$  are Majorana 2-spinors, with  $A = 1, \dots, \mathcal{N}$ , while the  $\Phi_i$  are real scalars with  $i = 2, \dots, \mathcal{N}$ . The Yukawa couplings are given by  $\rho_{AB}^i = \epsilon_{AB}$  for  $\mathcal{N} = 2$ , by (2.39) for  $\mathcal{N} = 4$ , and for  $\mathcal{N} = 8$  they are given by the matrices relating the  $\mathbf{8}_v$ ,  $\mathbf{8}_c$ , and  $\mathbf{8}_s$  representations of  $SO(8)$ , see (A.27). The set of such matrices is completed with the unit matrix, so that  $\rho_{AB}^C = \{\rho_{AB}^1 = \delta_{AB}, \rho_{AB}^i\}$ . All fields transform in the adjoint representation of  $SU(N)$ . The mode expansions are a slight generalization of (2.14)

$$\begin{aligned} \Phi_i &= \int \frac{d^2p}{(2\pi)^2} \frac{1}{\sqrt{2p^0}} \left( a_i^\dagger(p) e^{ip \cdot x} + a_i(p) e^{-ip \cdot x} \right), \\ A_\mu &= \int \frac{d^2p}{(2\pi)^2} \frac{1}{\sqrt{2p^0}} \epsilon_\mu(p, k) \left( a_1^\dagger(p) e^{ip \cdot x} + a_1(p) e^{-ip \cdot x} \right), \\ \lambda_A &= \int \frac{d^2p}{(2\pi)^2} \frac{1}{\sqrt{2p^0}} \left( u(p) \lambda_A^\dagger(p) e^{ip \cdot x} + u(p) \lambda_A(p) e^{-ip \cdot x} \right). \end{aligned} \quad (3.2)$$

We will be interested in calculating colour-ordered amplitudes. Labelling the four particles' gauge group indices as  $a_1, a_2, a_3$ , and  $a_4$ , one generically finds expressions depending upon the following contractions of the gauge group structure constants

$$\begin{aligned} f^{a_1 a_2} f^{a_3 a_4} &= -2 \text{Tr}[T^{a_1} T^{a_2} T^{a_3} T^{a_4}] + 2 \text{Tr}[T^{a_2} T^{a_1} T^{a_3} T^{a_4}] \\ &\quad + 2 \text{Tr}[T^{a_3} T^{a_1} T^{a_2} T^{a_4}] - 2 \text{Tr}[T^{a_3} T^{a_2} T^{a_1} T^{a_4}], \\ f^{a_1 a_4} f^{a_3 a_2} &= -2 \text{Tr}[T^{a_1} T^{a_2} T^{a_3} T^{a_4}] + 2 \text{Tr}[T^{a_4} T^{a_1} T^{a_3} T^{a_2}] \\ &\quad + 2 \text{Tr}[T^{a_3} T^{a_1} T^{a_4} T^{a_2}] - 2 \text{Tr}[T^{a_3} T^{a_2} T^{a_1} T^{a_4}], \\ f^{a_1 a_3} f^{a_2 a_4} &= -2 \text{Tr}[T^{a_1} T^{a_3} T^{a_2} T^{a_4}] + 2 \text{Tr}[T^{a_3} T^{a_1} T^{a_2} T^{a_4}] \\ &\quad + 2 \text{Tr}[T^{a_2} T^{a_1} T^{a_3} T^{a_4}] - 2 \text{Tr}[T^{a_2} T^{a_3} T^{a_1} T^{a_4}]. \end{aligned} \quad (3.3)$$

The colour-ordered contributions are those proportional to  $\text{Tr}[T^{a_1} T^{a_2} T^{a_3} T^{a_4}]$ , and so come from  $f^{a_1 a_2} f^{a_3 a_4}$  and  $f^{a_1 a_4} f^{a_3 a_2}$ . In what follows we have restored the more conventional counting of the coupling constant by rescaling all fields by  $g$ . We find it most convenient to present the colour-ordered amplitude at tree-level in the following way<sup>6</sup>

$$\left\langle \phi_{\mathcal{A}_1}^{a_1 \dagger}(p_1) \phi_{\mathcal{A}_2}^{a_2 \dagger}(p_2) \phi_{\mathcal{A}_3}^{a_3 \dagger}(p_3) \phi_{\mathcal{A}_4}^{a_4 \dagger}(p_4) \right\rangle = 2ig^2 \mathcal{C}(\phi_{\mathcal{A}_1} \phi_{\mathcal{A}_2} \phi_{\mathcal{A}_3} \phi_{\mathcal{A}_4}) \text{Tr}[T^{a_1} T^{a_2} T^{a_3} T^{a_4}] + \dots \quad (3.4)$$

<sup>5</sup>The action for the  $\mathcal{N} = 2$  ( $\mathcal{N} = 8$ ) theory is derived by dimensional reduction from the  $\mathcal{N} = 1$  theory in  $d = 4$  ( $d = 10$ ) in appendix A. The  $\mathcal{N} = 4$  theory is discussed in section 2.5.

<sup>6</sup>Note that the coupling  $g^2$  should be understood to be made dimensionless via the introduction of a renormalization scale  $\mu$ , so that  $g^2 \sim g^2/\mu$ .

where  $\phi_{\mathcal{A}}^\dagger(p) = \phi_{\mathcal{A}}^{a\dagger}(p)T^a$  is the creation operator for the given field  $\phi_{\mathcal{A}}$  as per the mode expansions given in (3.2),  $\mathcal{A}$  denotes a general flavour index, and the “...” refers to non-colour-ordered contributions. All momenta are taken to be in-going, so that  $\sum_{i=1}^4 p_i^\mu = 0$ .

### 3.1 Four fermion scattering

There are two Feynman diagrams contributing to the scattering of four  $\lambda_A^\dagger(p)$  external states; the gluon exchange and the scalar exchange. We may express these two contributions in terms of the following two expressions respectively

$$\begin{aligned}\mathcal{A}(1, 2, 3, 4) &\equiv \frac{\left(\bar{u}(p_1)\gamma^\mu u(p_2)\right)\left(\bar{u}(p_3)\gamma_\mu u(p_4)\right)}{(p_1 + p_2)^2}, \\ \mathcal{B}(1, 2, 3, 4) &\equiv \frac{\left(\bar{u}(p_1)u(p_2)\right)\left(\bar{u}(p_3)u(p_4)\right)}{(p_1 + p_2)^2}.\end{aligned}\tag{3.5}$$

Note that  $\mathcal{A}(1, 2, 3, 4) = \mathcal{A}(2, 1, 3, 4) = \mathcal{A}(1, 2, 4, 3) = \mathcal{A}(2, 1, 4, 3)$ , while  $\mathcal{B}(1, 2, 3, 4) = -\mathcal{B}(2, 1, 3, 4) = -\mathcal{B}(1, 2, 4, 3) = \mathcal{B}(2, 1, 4, 3)$ . We find

$$\begin{aligned}\mathcal{C}(\lambda_{A_1}\lambda_{A_2}\lambda_{A_3}\lambda_{A_4}) &= \delta_{A_1 A_2}\delta_{A_3 A_4}\left(\mathcal{B}(4, 1, 2, 3) + \mathcal{A}(1, 2, 3, 4)\right) \\ &\quad - \delta_{A_1 A_3}\delta_{A_2 A_4}\left(\mathcal{B}(4, 1, 2, 3) - \mathcal{B}(1, 2, 3, 4)\right) \\ &\quad - \delta_{A_1 A_4}\delta_{A_2 A_3}\left(\mathcal{A}(4, 1, 2, 3) + \mathcal{B}(1, 2, 3, 4)\right).\end{aligned}\tag{3.6}$$

We note that the amplitude is manifestly  $SO(\mathcal{N})$  invariant.

### 3.2 Two boson - two fermion tree-level amplitudes

We begin by calculating the amplitudes

$$\begin{aligned}&\left\langle a_{i_1}^{a_1\dagger}(p_1) a_{i_2}^{a_2\dagger}(p_2) \lambda_{A_3}^{a_3\dagger}(p_3) \lambda_{A_4}^{a_4\dagger}(p_4) \right\rangle, \\ &\left\langle a_1^{a_1\dagger}(p_1) a_{i_2}^{a_2\dagger}(p_2) \lambda_{A_3}^{a_3\dagger}(p_3) \lambda_{A_4}^{a_4\dagger}(p_4) \right\rangle, \\ &\left\langle a_{i_1}^{a_1\dagger}(p_1) a_1^{a_2\dagger}(p_2) \lambda_{A_3}^{a_3\dagger}(p_3) \lambda_{A_4}^{a_4\dagger}(p_4) \right\rangle, \\ &\left\langle a_1^{a_1\dagger}(p_1) a_1^{a_2\dagger}(p_2) \lambda_{A_3}^{a_3\dagger}(p_3) \lambda_{A_4}^{a_4\dagger}(p_4) \right\rangle.\end{aligned}\tag{3.7}$$

There are contributions from a fermion exchange

$$\begin{aligned}\mathcal{C}_F(a_1 a_1 \lambda_{A_3} \lambda_{A_4}) &= -\delta_{A_3 A_4} \bar{u}(p_4) \not{\epsilon}(p_1) \frac{(\not{p}_2 + \not{p}_3)}{(p_2 + p_3)^2} \not{\epsilon}(p_2) u(p_3), \\ \mathcal{C}_F(a_{i_1} a_{i_2} \lambda_{A_3} \lambda_{A_4}) &= \left((\rho^{i_1})^T \rho^{i_2}\right)_{A_4 A_3} \bar{u}(p_4) \frac{(\not{p}_2 + \not{p}_3)}{(p_2 + p_3)^2} u(p_3), \\ \mathcal{C}_F(a_{i_1} a_1 \lambda_{A_3} \lambda_{A_4}) &= \rho_{A_3 A_4}^{i_1} \bar{u}(p_4) \frac{(\not{p}_2 + \not{p}_3)}{(p_2 + p_3)^2} \not{\epsilon}(p_2) u(p_3), \\ \mathcal{C}_F(a_1 a_{i_2} \lambda_{A_3} \lambda_{A_4}) &= \rho_{A_3 A_4}^{i_2} \bar{u}(p_4) \not{\epsilon}(p_1) \frac{(\not{p}_2 + \not{p}_3)}{(p_2 + p_3)^2} u(p_3),\end{aligned}\tag{3.8}$$

and a boson exchange,

$$\begin{aligned}
\mathcal{C}_B(a_1 a_1 \lambda_{A_3} \lambda_{A_4}) &= -\delta_{A_3 A_4} \left[ -2p_1 \cdot \epsilon(p_2) \frac{\bar{u}(p_4) \not{\epsilon}(p_1) u(p_3)}{(p_1 + p_2)^2} + 2p_2 \cdot \epsilon(p_1) \frac{\bar{u}(p_4) \not{\epsilon}(p_2) u(p_3)}{(p_1 + p_2)^2} \right. \\
&\quad \left. + \epsilon(p_1) \cdot \epsilon(p_2) \frac{\bar{u}(p_4) (\not{p}_1 - \not{p}_2) u(p_3)}{(p_1 + p_2)^2} \right], \\
\mathcal{C}_B(a_{i_1} a_{i_2} \lambda_{A_3} \lambda_{A_4}) &= -\delta_{i_1 i_2} \delta_{A_3 A_4} \frac{\bar{u}(p_4) (\not{p}_1 - \not{p}_2) u(p_3)}{(p_1 + p_2)^2}, \\
\mathcal{C}_B(a_{i_1} a_1 \lambda_{A_3} \lambda_{A_4}) &= -2\rho_{AB}^{i_1} (p_1 \cdot \epsilon(p_2)) \frac{\bar{u}(p_4) u(p_3)}{(p_1 + p_2)^2}, \\
\mathcal{C}_B(a_1 a_{i_2} \lambda_{A_3} \lambda_{A_4}) &= 2\rho_{AB}^{i_2} (p_2 \cdot \epsilon(p_1)) \frac{\bar{u}(p_4) u(p_3)}{(p_1 + p_2)^2}.
\end{aligned} \tag{3.9}$$

The complete tree-level amplitudes are obtained by taking the sum of the boson and fermion exchanges.

One may determine the remaining amplitudes as follows

$$\begin{aligned}
\mathcal{C}(a_D \lambda_A \lambda_B a_C) &= \mathcal{C}(a_C a_D \lambda_A \lambda_B) \text{ with } p_1 \rightarrow p_4, p_2 \rightarrow p_1, p_3 \rightarrow p_2, p_4 \rightarrow p_3, \\
\mathcal{C}(a_C \lambda_A a_D \lambda_B) &= -\mathcal{C}(a_C a_D \lambda_A \lambda_B) \text{ with } p_2 \leftrightarrow p_3 - \mathcal{C}(a_C \lambda_A \lambda_B a_D) \text{ with } p_3 \leftrightarrow p_4,
\end{aligned} \tag{3.10}$$

where  $a_C$  indicates either a scalar or a gauge field, see (3.12). The amplitudes with a fermion in the first position may be determined through applications of similar rules starting with

$$\mathcal{C}(\lambda_D \lambda_C a_B a_A) = \mathcal{C}(a_A a_B \lambda_C \lambda_D) \text{ with } p_1 \leftrightarrow p_4, p_2 \leftrightarrow p_3. \tag{3.11}$$

### 3.3 Four boson tree-level amplitudes

The four boson amplitude stems from a boson exchange diagram and a contact diagram stemming from the 4-boson vertices. We find that the results may be compactly expressed by enlarging the index  $i$  on the scalar field to include a first component which is identified with the gauge field degree of freedom, i.e.

$$a_A^\dagger = (a_1^\dagger, a_i^\dagger). \tag{3.12}$$

The color-ordered amplitude is then read-off from the following compact expression

$$\begin{aligned}
\mathcal{C}(a_{A_1} a_{A_2} a_{A_3} a_{A_4}) &\rightarrow \frac{1}{(p_1 + p_2)^2} \left[ \Theta(1, 2) \cdot \Theta(3, 4) + \frac{(p_1 + p_2)^2}{2} \mathcal{F}_{MN}(1, 2) \mathcal{F}^{MN}(3, 4) \right] \\
&\quad + \frac{1}{(p_1 + p_4)^2} \left[ \Theta(1, 4) \cdot \Theta(3, 2) + \frac{(p_1 + p_4)^2}{2} \mathcal{F}_{MN}(1, 4) \mathcal{F}^{MN}(3, 2) \right],
\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}\Theta_M(1, 2) &\equiv 2p_2 \cdot \epsilon(p_1) \epsilon_M(p_2) - 2p_1 \cdot \epsilon(p_2) \epsilon_M(p_1) + (p_1 - p_2)_M \epsilon(p_1) \cdot \epsilon(p_2), \\ \mathcal{F}_{MN}(1, 2) &\equiv \epsilon_M(p_1) \epsilon_N(p_2) - \epsilon_N(p_1) \epsilon_M(p_2),\end{aligned}\tag{3.14}$$

and where  $M$  and  $N$  are 4-dimensional (for the  $\mathcal{N} = 2$  case), and 10-dimensional (for the  $\mathcal{N} = 8$  case) indices for which

$$\epsilon^M(p) = \begin{cases} (\epsilon^\mu(p), 0, \dots, 0), & a^\dagger(p) \\ (0, \dots, 0, \underbrace{1, 0, \dots, 0}_{i+1}), & a_i^\dagger(p) \end{cases} \quad p^M = (p^\mu, 0, \dots, 0).\tag{3.15}$$

### 3.4 Manifestly $SO(\mathcal{N})$ invariant forms for the amplitudes

Using the spinor formalism developed in section 2, we find that the amplitudes may be presented in a way which shows manifest  $SO(\mathcal{N})$  invariance. Using (3.12), we find the following expressions

$$\begin{aligned}\mathcal{C}(a_{A_1} a_{A_2} a_{A_3} a_{A_4}) &= -2\delta_{A_1 A_2} \delta_{A_3 A_4} \frac{\langle 13 \rangle \langle 24 \rangle}{\langle 12 \rangle \langle 34 \rangle} + 2\delta_{A_1 A_3} \delta_{A_2 A_4} + 2\delta_{A_1 A_4} \delta_{A_2 A_3} \frac{\langle 13 \rangle \langle 24 \rangle}{\langle 23 \rangle \langle 41 \rangle}, \\ \mathcal{C}(\lambda_{A_1} \lambda_{A_2} \lambda_{A_3} \lambda_{A_4}) &= 2\delta_{A_1 A_2} \delta_{A_3 A_4} \frac{\langle 13 \rangle^2 \langle 23 \rangle}{\langle 12 \rangle^2 \langle 41 \rangle} - 2\delta_{A_1 A_3} \delta_{A_2 A_4} \frac{\langle 34 \rangle}{\langle 12 \rangle} - 2\delta_{A_1 A_4} \delta_{A_2 A_3} \frac{\langle 13 \rangle^2 \langle 12 \rangle}{\langle 14 \rangle^2 \langle 34 \rangle},\end{aligned}\tag{3.16}$$

and, defining  $\rho^{A_1 A_2} \equiv \frac{1}{2} \left( (\rho^{A_1})^T \rho^{A_2} - (\rho^{A_2})^T \rho^{A_1} \right)$ ,

$$\begin{aligned}\mathcal{C}(a_{A_1} a_{A_2} \lambda_{A_3} \lambda_{A_4}) &= -\delta_{A_1 A_2} \delta_{A_3 A_4} \frac{\langle 13 \rangle^2}{\langle 12 \rangle^2} \left( \frac{\langle 13 \rangle}{\langle 14 \rangle} + \frac{\langle 23 \rangle}{\langle 24 \rangle} \right) + \left( \rho^{A_1 A_2} \right)_{A_3 A_4} \frac{\langle 31 \rangle}{\langle 14 \rangle}, \\ \mathcal{C}(a_{A_1} \lambda_{A_2} \lambda_{A_3} a_{A_4}) &= -\delta_{A_1 A_4} \delta_{A_2 A_3} \frac{\langle 42 \rangle^2}{\langle 41 \rangle^2} \left( \frac{\langle 42 \rangle}{\langle 43 \rangle} + \frac{\langle 12 \rangle}{\langle 13 \rangle} \right) - \left( \rho^{A_1 A_4} \right)_{A_2 A_3} \frac{\langle 24 \rangle}{\langle 43 \rangle}, \\ \mathcal{C}(a_{A_1} \lambda_{A_2} a_{A_3} \lambda_{A_4}) &= -\delta_{A_1 A_3} \delta_{A_2 A_4} \left( \frac{\langle 12 \rangle}{\langle 14 \rangle} - \frac{\langle 23 \rangle}{\langle 34 \rangle} \right) \\ &\quad + \left( \rho^{A_1 A_3} \right)_{A_2 A_4} \left( \frac{\langle 12 \rangle}{\langle 14 \rangle} + \frac{\langle 23 \rangle}{\langle 34 \rangle} \right).\end{aligned}\tag{3.17}$$

Note that the mixed amplitudes with a fermion in the first position can be obtained straightforwardly using (3.10) and (3.11). The invariant expressions (3.16) and (3.17) can straightforwardly be checked to satisfy the associated supersymmetry algebra given in (A.28). As an explicit example, we consider the  $\mathcal{N} = 2$  case in the next subsection.

### 3.5 Reconstruction of helicity

Using the  $\mathcal{N} = 2$  theory as an example, we show how the  $d = 4$  MHV amplitudes are recovered. Following section 2.2, one may define a three-dimensional analogue of



helicity

$$\begin{aligned} a_{\pm} &= \frac{1}{\sqrt{2}} (a_1 \pm i a_2), \\ \lambda_{\pm} &= \frac{1}{\sqrt{2}} (\lambda_1 \pm i \lambda_2), \end{aligned} \tag{3.18}$$

from which one finds  $\mathcal{C}(\phi_+ \phi_+ \phi_+ \phi_+) = \mathcal{C}(\phi_- \phi_- \phi_- \phi_-) = 0$  for all fields. The non-zero amplitudes are as follows

$$\mathcal{C}_{\text{MHV}}(aaaa) = 2 \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}, \tag{3.19}$$

where  $i, j$  denote the positions of the negative (assuming the other two are positive) or positive (assuming the other two are negative) helicity states. This is of course the Parke-Taylor formula. One also finds

$$\begin{aligned} \mathcal{C}(\lambda_+ \lambda_+ \lambda_- \lambda_-) &= \frac{\langle 13 \rangle}{\langle 24 \rangle} \mathcal{C}(a_+ a_+ a_- a_-), \\ \mathcal{C}(\lambda_+ \lambda_- \lambda_+ \lambda_-) &= \frac{\langle 24 \rangle}{\langle 13 \rangle} \mathcal{C}(a_+ a_- a_+ a_-), \\ \mathcal{C}(\lambda_+ \lambda_- \lambda_- \lambda_+) &= \frac{\langle 13 \rangle}{\langle 24 \rangle} \mathcal{C}(a_+ a_- a_- a_+), \end{aligned} \tag{3.20}$$

in agreement with (2.32). The non-zero mixed fermion-boson amplitudes are in agreement with (2.31).

## 4 Comments on loop corrections for $\mathcal{N} = 8$

We briefly comment on the integrands corresponding to the one-loop correction to the tree-level amplitudes using the case of  $\mathcal{N} = 8$  theory as an example. The method of unitarity cuts allows us to efficiently evaluate the one-loop contribution to the four-particle  $\mathcal{N} = 8$  amplitudes from the knowledge of the corresponding tree-level quantities. Furthermore, since the tree-level amplitudes are nothing but the four-dimensional ones evaluated in a boosted frame where  $k_3 = 0$  (c.f. section 2.3), the integrands contributing to the loop corrections to any amplitude can be easily constructed from the known results in four dimensions. For, instance, consider a four-particle amplitude  $\mathcal{M}(1^{h_1}, 2^{h_2}, 3^{h_3}, 4^{h_4})$ . In  $\mathcal{N} = 4$  SYM in  $d = 4$ , the contribution from the  $s$  or  $t$  channel cut to this amplitude is generically of the form

$$\begin{aligned} \mathcal{M}(1^{h_1}, 2^{h_2}, 3^{h_3}, 4^{h_4})|_{\text{cut}} &= \sum_{h, h'} \int \frac{d^4 k}{(2\pi)^4} 2\pi \delta^+(l_1^2) \delta^+(l_2^2) \\ &\times \mathcal{M}_1^{\text{tree}}(-l_1^h, 1^{h_1}, 2^{h_2}, l_2^{h'}) \mathcal{M}_2^{\text{tree}}(-l_2^{\bar{h}'}, 3^{h_3}, 4^{h_4}, l_1^{\bar{h}}), \end{aligned} \tag{4.1}$$

where  $\mathcal{M}_i$  are the amplitudes that contribute to the particular cut,  $l_1 = k$ , and  $l_2 = k - p_1 - p_2$ , where  $k$  is the loop-momentum. One replaces the delta functions by  $\frac{i}{2\pi l_{i=1,2}^2}$  to construct the full integrand of the Feynman integral contributing to the one-loop correction to  $\mathcal{M}$ . Given the relations between the amplitudes of the three and

four-dimensional theories, it readily follows that the corresponding three-dimensional amplitude is given by

$$\begin{aligned} \mathcal{M}_{d=3}(1^{h_1}, 2^{h_2}, 3^{h_3}, 4^{h_4})|_{cut} &= \sum_{h,h'} \int \frac{d^3k}{(2\pi)^3} 2\pi \delta^+(l_1^2) \delta^+(l_2^2) \\ &\times \mathcal{M}_{d=3,1}^{tree}(-l_1^h, 1^{h_1}, 2^{h_2}, l_2^{h'}) \mathcal{M}_{d=3,2}^{tree}(-l_2^{\bar{h}'}, 3^{h_3}, 4^{h_4}, l_1^{\bar{h}}), \end{aligned} \quad (4.2)$$

where,  $\mathcal{M}_{d=3}$  are obtained setting the fourth components of all momenta to zero in the corresponding four-dimensional quantity. In the three-dimensional context,  $h_i$  corresponds to  $U(1)$  charge carried by the states. All the algebraic identities between various spinor products that are used to bring the integrands of the one-loop amplitudes in  $\mathcal{N} = 4$  SYM to a scalar-box integral continue to hold after the dimensional reduction to  $d = 3$  as well.

Since there is only one-independent four-particle amplitude - the rest are related to any given amplitude by the constraints of supersymmetry - we only give the answer for the one-loop ‘‘MHV’’ amplitude in three dimensions. After accounting for the  $t$  and  $s$  channel cuts, one has

$$\langle a_+ a_+ a_- a_- \rangle_1 = -st \langle a_+ a_+ a_- a_- \rangle_0 I \quad (4.3)$$

The subscripts (0,1) refer to tree-level and one-loop respectively, while  $I$  is the three-dimensional massless scalar box integral

$$I = \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2(q+p_1)^2(q+p_1+p_2)^2(q-p_4)^2}. \quad (4.4)$$

The massless scalar box integral is IR divergent in dimensions  $d \leq 4$ , and so is far away from convergence in  $d = 3$ . A potential method for regulating it is to use the Coulomb branch as has been done for  $\mathcal{N} = 4$  SYM in  $d = 4$  in [31]. We leave this issue to a further publication.

Although we concentrated on the  $\mathcal{N} = 8$  theory here, the  $d = 3$  theories with lower SUSY also share the property that the one-loop integrands can be gotten straightforwardly through dimensional reduction of the appropriate theory in four dimensions, and therefore benefit from the application of unitarity-based methods employed there.

## 5 Relation to SCS theories: D2 vs. M2-brane $S$ -matrices

In this section, we comment on the realization of the  $SO(\mathcal{N})$  symmetric on-shell supersymmetry algebra for  $\mathcal{N} \geq 4$  SCS theories. In particular we focus on the  $\mathcal{N} = 8$  BLG theory [21, 22, 23, 24], to which the SYM theory with  $SU(2)$  gauge group is expected to flow in the deep IR. We make an explicit identification between the on-shell degrees of freedom of the two theories and show that the ‘‘matrix’’ part of the four-particle scattering matrix  $S_{ijkl}(\{\mathcal{W}\}; t, s)$  of (1.1) is the same for both theories. Generally, one expects the  $SO(8)$  symmetry to be manifest for all observables of the SYM theory only at  $g_{YM}^2 \rightarrow \infty$ . However, since we have determined  $R$ -symmetry structure of the

four-particle amplitude to all orders in perturbation theory, we are able to compare the manifestly  $SO(8)$  symmetric observables in both the theories in a transparent manner.

In the case of supersymmetric Chern-Simons theories, the construction of Gaiotto and Witten [32] and its generalizations [33, 34] allow one to construct  $\mathcal{N} \geq 4$  SCS theories in a unified manner. One starts with a symplectic group  $Sp(2n)$ , which contains the gauge group  $G$  as a subgroup.  $Sp(2n)$  has an antisymmetric form  $\omega_{AB}$  and a Cartan metric  $k^{mn}$ . The generators of the gauge group  $t_B^{mA}$  are  $2n \times 2n$  matrices for each value of  $m$ .  $m$  should be regarded as an “adjoint” index, while  $A, B$  can be thought of as “fundamental” indices. The gauge potential  $A_\mu^m$  has a Lorentz and an adjoint index as expected. The matter fields (bosons)  $q_\beta^A$  and (fermions)  $\psi_\beta^A$  carry two different  $SU(2)$  (dotted and undotted) indices, apart from the index corresponding to gauge group  $A$ . The matter fields are taken to satisfy the reality conditions

$$q_{A\alpha}^\dagger = \epsilon_{\alpha\beta}\omega_{AB}q_\beta^B, \quad \psi_{A\dot{\alpha}}^\dagger = \epsilon_{\dot{\alpha}\dot{\beta}}\omega_{AB}\psi_\beta^B. \quad (5.1)$$

The conditions for  $\mathcal{N} = 4$  SUSY were derived by Gaiotto and Witten to be

$$k_{mn}t_{a(b}^mt_{CD)}^n = 0, \quad (5.2)$$

where the brackets denote symmetrization of the indices. The supersymmetry generators act on the asymptotic states as follows [26]

$$\mathcal{Q}_{a\alpha\dot{\beta}}|q_\gamma\rangle = u_a\epsilon_{\alpha\gamma}|\psi_\beta\rangle, \quad \mathcal{Q}_{a\alpha\dot{\beta}}|\psi_\gamma\rangle = u_a\epsilon_{\dot{\beta}\dot{\gamma}}|q_\alpha\rangle. \quad (5.3)$$

The generators close as

$$\{\mathcal{Q}_{a\alpha\dot{\beta}}, \mathcal{Q}_{b\gamma\dot{\delta}}\} = \epsilon_{\alpha\gamma}\epsilon_{\dot{\beta}\dot{\delta}}P_{ab}. \quad (5.4)$$

One can add more matter multiplets, the so called twisted hypermultiplets, (bosons)  $\tilde{q}_\alpha^A$  and (fermions)  $\tilde{\psi}_\alpha^A$ , transforming under a representation of the gauge group generated by  $\tilde{t}_{AB}^m$  which in general  $\neq t_{AB}^m$ . The twisting refers to the interchanging of the  $SU(2)$  indices for the bose and fermi particles with respect to the original matter fields. At this level of generality one can only have  $\mathcal{N} = 4$  SUSY. It was observed in [33, 34] that, if  $t = \tilde{t}$ , then one has an enhancement of SUSY to  $\mathcal{N} = 5$ . Furthermore if  $t = \tilde{t}$  can be decomposed as  $(\mathcal{R}, \bar{\mathcal{R}})$ , e.g. as in bifundamentals  $(N, \bar{N})$  of  $SU(N)$ , then one has enhancement to  $\mathcal{N} = 6$  SUSY. Finally, if the representations are real,  $\mathcal{R} = \bar{\mathcal{R}}$ , then one recovers  $\mathcal{N} = 8$  supersymmetry. For the last case, the only known example is the  $SU(2)$  superconformal Chern-Simons theory of BLG; we can make an explicit identification between the degrees of freedom of the Yang-Mills theory and the  $\mathcal{N} = 8$  SCS theory it is expected to be described by at infinite coupling. In this case, one has eight real scalars  $X^{A(I)}$ . The  $SU(2)$  indices of the preceding discussion have been promoted to an  $SO(8)$  index  $I$ . The supersymmetry variation of the scalars is given by

$$\delta X^{A(I)} = i\bar{\epsilon}\Gamma^{I+2}\Psi^A. \quad (5.5)$$

For the SYM theory, the variation of the seven scalars obtained by the dimensional reduction of the  $\mathcal{N} = 1, d = 10$  theory  $\Phi^{I=3\cdots 9}$  is given in the three-dimensional notation by

$$\delta\Phi^{A(I)} = i\bar{\epsilon}\Gamma^I\Psi^A. \quad (5.6)$$

The “gauge” indices of the scalars of the BLG theory correspond to a 3-algebra (which for the BLG theory is an  $SU(2)$  algebra in disguise [35]), while for the SYM theory

they are the adjoint  $SU(2)$  indices. However, since we are only interested in the color-ordered amplitudes, we can disregard the gauge index and immediately see that the on-shell supersymmetry variations of the seven scalars of the SYM theory coincide with the variations of  $X^{I=1\dots 7}$ . However,

$$\delta X^{A(8)} = i\bar{\epsilon}\Gamma^{11}\Psi^A = i\bar{\epsilon}\Psi^A, \quad (5.7)$$

where we have used the  $d = 10$  Weyl condition. The on-shell version of this transformation is

$$(Q_A)_\alpha |X^8\rangle = \frac{1}{2}u_\alpha |\lambda_A\rangle, \quad (5.8)$$

which, as we have seen previously is exactly the same as the transformation law for  $a_1$ , the scalar that is dual to the gauge field of the Yang-Mills theory. Thus, for the on-shell supersymmetry algebra  $X_{BLG}^I \leftrightarrow (\Phi^I, A_\mu)_{SYM}$ . The rest of the on-shell SUSY algebra is uniquely determined by the requirements of the closure of the generators on to  $P$ . Thus, the full on-shell algebras of the  $\mathcal{N} = 8$  SCS and SYM theories are the same. This immediately implies that the “matrix” structures of the  $S$ -matrices of the two theories are also identical.  $S_{ijkl}(\{\mathcal{W}\}; t, s)$  for the SYM theory can be readily extracted from the expressions in section 3.4 which is the same as the corresponding quantity for the SCS theory found in [26]. Thus the flow of the D2-brane theory to the M2-brane model (at the level of four-particle amplitudes) corresponds to understanding how the one independent amplitude which takes on the MHV form (3.19) at extreme weak coupling, flows to the four-boson scattering amplitude given in [26], at infinite coupling. Obviously, obtaining the full interpolating form of the four-particle amplitude would be an enormous progress towards establishing the connection between D2 and M2-brane theories.

## 6 Concluding remarks

We have presented an on-shell formalism that reveals several algebraic properties of  $S$ -matrices of three-dimensional Yang-Mills theories that are not evident at the level of the corresponding Lagrangians. In particular our framework uncovers a hidden  $U(1)$  symmetry, which is an augmentation of the helicity degree of freedom of four-dimensional parent theories, to a continuous symmetry upon dimensional reduction. This emergent  $U(1)$  lifts the  $SO(\mathcal{N} - 1)$  symmetry of the gauge theory Lagrangians to an  $SO(\mathcal{N})$  symmetry of the on-shell algebra and  $S$ -matrix. We have been able to confirm that the manifest  $SO(\mathcal{N})$  invariance is indeed realized by four-particle amplitudes to all orders in perturbation theory while presenting arguments in favor of the same phenomenon for higher-particle amplitudes. As an application of the methodology presented in this work, we used the manifest  $SO(8)$  invariance of the four-particle amplitude of the  $\mathcal{N} = 8$  theory to show that the amplitude of the SYM theory is the same as that of the BLG theory up to a single function. We have also presented detailed results for the tree-level forms of the four-particle amplitudes of all the gauge theories considered in this paper while paying special attention to the  $SO(\mathcal{N})$  invariance of the results. Other than the issue of symmetry enhancement, we have also shown that recursion relations for  $d = 3$  SYM theories can readily be obtained via dimensional reduction.

The results of this paper point to several exciting directions for future research. At tree-level, it is a simple exercise to obtain three-dimensional results via dimensional reduction from four dimensions. Indeed, even at the loop-level, it would appear that the  $d = 3$  integrands are trivial generalizations of their  $d = 4$  counterparts, and in this sense are determined by the structure of the parent theory, e.g. for  $\mathcal{N} = 4$  in  $d = 4$ , this structure is believed to be captured by a Grassmannian [36]. The structure of IR divergences, however, may be very different in three dimensions. The question of how the various symmetries and formulations of four-dimensional amplitudes translate to three dimensions once internal momenta have been integrated over is a very interesting question, and one which we hope to report on in the near future.

Finally, it would be extremely interesting to build on the connection between the  $S$ -matrices of M2 and D2-brane theories pointed out in this paper. At the level of four-particle amplitudes, we have reduced the problem of understanding the flow of  $\mathcal{N} = 8$  SYM theory to the BLG theory to understanding the asymptotic behavior of a single function. Given the recent hints of the existence of a twistorial structure and Yangian symmetries for the scattering matrices of SCS theories [27, 28, 29] it is perhaps not inconceivable that the relationship between the SCS and SYM theories can be understood very concretely at the on-shell level by uncovering the corresponding algebraic structures for the SYM amplitudes.

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## A Dimensional reductions

### A.1 Conventions

We work in mostly positive signature. Our fermions are Majorana and obey  $\bar{\lambda}_A = \lambda_A^T C_d$ , where  $C_d$  is the  $d$ -dimensional charge conjugation matrix which is identified with the zero component gamma matrix. Our scalars are real. Each field in the theory  $\phi = \phi^a T^a$ ,  $a = 1, \dots, N^2 - 1$ , where the generators  $T^a$  of  $SU(N)$  are  $N \times N$  matrices obeying the following identities

$$\begin{aligned} T^a T^a &= \frac{N^2 - 1}{2N} \mathbf{1}, \quad \text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}, \quad [T^a, T^b] = i f^{abc} T^c, \quad f^{abc} f^{abd} = N \delta^{cd}, \\ \{T^a, T^b\} &= \frac{1}{N} \delta^{ab} \mathbf{1} + d^{abc} T^c, \end{aligned} \quad (\text{A.1})$$

and the gauge covariant derivative is defined as

$$\begin{aligned} D_\mu \phi &= \partial_\mu \phi - i[A_\mu, \phi], \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]. \end{aligned} \quad (\text{A.2})$$

The free field propagators for the various fields are given by

$$\begin{aligned} \langle \Phi_i^a(p) \Phi_j^b(-p) \rangle &= -\delta_{ij} \frac{ig^2 \delta^{ab}}{p^2}, \quad \langle A_\mu^a(p) A_\nu^b(-p) \rangle = -\delta^{ab} \frac{ig^2 \eta_{\mu\nu}}{p^2}, \\ \langle \lambda_{A\alpha}^a(p) \lambda_{B\beta}^b(-p) \rangle &= -\delta_{AB} \delta^{ab} \frac{ig^2 p_\mu (\gamma^\mu C_3^{-1})_{\alpha\beta}}{p^2}. \end{aligned} \quad (\text{A.3})$$

We have chosen Feynman gauge for the gauge field. The ghost action is not given as we are working at tree-level.

## A.2 Spinor identities

Defining  $\langle p| \equiv \bar{u}(p)$  and  $|p\rangle \equiv u(p)$ , we note the following relations

$$\begin{aligned}
\langle ij\rangle &= -\langle ji\rangle, \\
\not{\epsilon}(p_1, p_2)u(p_3) &= -2\frac{\langle 13\rangle}{\langle 12\rangle}u(p_2) + u(p_3), \\
\bar{u}(p_3)\not{\epsilon}(p_1, p_2) &= -2\frac{\langle 32\rangle}{\langle 12\rangle}\bar{u}(p_1) + \bar{u}(p_3), \\
p_3 \cdot \not{\epsilon}(p_1, p_2) &= 2(p_1 \cdot p_3) \frac{\langle 23\rangle}{\langle 21\rangle\langle 31\rangle}, \\
\langle 1|\gamma^\mu|2\rangle\langle 3|\gamma_\mu|4\rangle &= \langle 13\rangle\langle 42\rangle + \langle 23\rangle\langle 41\rangle, \\
\langle 12\rangle\langle 34\rangle &= \langle 23\rangle\langle 41\rangle - \langle 13\rangle\langle 42\rangle, \\
p_i \cdot p_j &= -\frac{\langle ij\rangle^2}{2}, \\
\frac{\langle 14\rangle}{\langle 23\rangle} &= -\frac{\langle 24\rangle}{\langle 13\rangle}, \quad \frac{\langle 12\rangle}{\langle 34\rangle} = -\frac{\langle 13\rangle}{\langle 24\rangle}.
\end{aligned} \tag{A.4}$$

The last two relations follow from momentum conservation for 4-particle scattering.

## A.3 $\mathcal{N} = 2$ theory

We begin with  $\mathcal{N} = 1$  SYM in  $d = 4$ , whose action is

$$S_{\mathcal{N}=1, d=4} = \frac{1}{g^2} \text{Tr} \int d^4x \left( -\frac{1}{2} F_{MN} F^{MN} + i\bar{\Psi} \Gamma^M D_M \Psi \right), \tag{A.5}$$

where  $M, N = 0, \dots, 3$  and we are using mostly plus signature

$$\eta_{MN} = \text{diag}(-1, 1, 1, 1), \tag{A.6}$$

and  $\Psi$  is a four-component Majorana spinor, and we use the real representation of the gamma-matrices provided in (2.8). The charge conjugation matrix is  $C_4 = \Gamma^0$  and  $\bar{\Psi} \equiv \Psi^T C_4$ . Let us write the Majorana spinor  $\Psi$  as follows

$$\Psi = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}. \tag{A.7}$$

Under dimensional reduction whereby we eliminate the last dimension, i.e.  $\partial_3 \rightarrow 0$ , we obtain the following action

$$S_{\mathcal{N}=2, d=3} = \frac{1}{g^2} \text{Tr} \int d^3x \left( -\frac{1}{2} F_{MN} F^{MN} + i\bar{\lambda}_A \gamma^\mu D_\mu \lambda_A + \epsilon_{AB} \bar{\lambda}_A [\Phi, \lambda_B] \right), \tag{A.8}$$

where  $A = 1, 2$ ,  $\mu = 0, 1, 2$ ,  $\epsilon_{12} = +1$ , and where the real representation of the three-dimensional gamma matrices used is

$$\gamma^\mu = (i\sigma^2, \sigma^1, \sigma^3), \quad C_3 = i\sigma^2, \quad \eta_{\mu\nu} = \text{diag}(-1, 1, 1), \tag{A.9}$$

and  $\bar{\lambda} \equiv \lambda^T C_3$ . Note that we have introduced

$$\Phi \equiv A_3, \quad (\text{A.10})$$

and assumed that  $\partial_3 = 0$  in the kinetic term for the gauge fields.

### A.3.1 Supersymmetry

We note the  $\mathcal{N} = 2$  SUSY transformations. Those of the original action (A.5) are

$$\delta A_N = -2i\bar{\Psi}\Gamma_N\epsilon, \quad \delta\Psi = F^{MN}\Gamma_{MN}\epsilon. \quad (\text{A.11})$$

We decompose  $\epsilon$  in terms of  $d = 3$  SUSY parameters

$$\epsilon = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}. \quad (\text{A.12})$$

We find

$$\begin{aligned} \delta A_\mu &= -2i\bar{\lambda}_A\gamma_\mu\eta_A, \\ \delta\Phi &= -2i\epsilon_{AB}\bar{\lambda}_A\eta_B, \\ \delta\lambda_A &= F_{\mu\nu}\gamma^{\mu\nu}\eta_A + 2\partial_\mu\Phi\epsilon_{AB}\gamma^\mu\eta_B. \end{aligned} \quad (\text{A.13})$$

From the standard mode expansions (2.14) we have

$$a_1^\dagger = \epsilon_\mu(p)A^\mu(p), \quad a_2^\dagger = \Phi(p), \quad \lambda_A^\dagger = \frac{1}{2p^0}u(p)\lambda_A(p). \quad (\text{A.14})$$

Note that  $u(p)u(p) = 2p^0$ . In appendix B, the supercharge is calculated in the four-dimensional formalism. Plugging in the mode expansions and using

$$u(p)\not{p} = -2p^0\bar{u}(p), \quad (\text{A.15})$$

and

$$\bar{u}\not{\epsilon} = -\bar{u}, \quad (\text{A.16})$$

which follows from (2.13), one recovers (2.15) and (2.16).

## A.4 $\mathcal{N} = 8$ theory

The 10-dimensional gamma matrices, in mostly positive signature, may be expressed using (2.8), as follows

$$\tilde{\Gamma}^M = \gamma^M \otimes \mathbb{1}_8, \quad \tilde{\Gamma}^I = i\gamma^{0123} \otimes \eta^I. \quad (\text{A.17})$$

where the  $SO(6)$  gamma matrices  $\eta^I$  are given by

$$\begin{aligned} \eta^I = \{ & \sigma^2 \otimes \sigma^2 \otimes \sigma^1, -\sigma^2 \otimes \sigma^2 \otimes \sigma^3, \sigma^2 \otimes \mathbb{1} \otimes \sigma^2, \\ & -\sigma^1 \otimes \sigma^1 \otimes \sigma^2, -\sigma^1 \otimes \sigma^2 \otimes \mathbb{1}, \sigma^1 \otimes \sigma^3 \otimes \sigma^2 \}. \end{aligned} \quad (\text{A.18})$$

The 6-dimensional charge conjugation matrix is

$$C_6 = \sigma^1 \otimes \mathbb{1} \otimes \mathbb{1}, \quad (\text{A.19})$$

while

$$\eta^{123456} = i\sigma^3 \otimes \mathbb{1} \otimes \mathbb{1}. \quad (\text{A.20})$$

This gives

$$\begin{aligned} \tilde{F}^{11} &= \tilde{F}^{0123456789} = i \begin{pmatrix} 0 & \mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{pmatrix} \otimes \begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix}, \\ C_{10} &= \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix} \otimes \begin{pmatrix} 0 & \mathbb{1}_4 \\ \mathbb{1}_4 & 0 \end{pmatrix}. \end{aligned} \quad (\text{A.21})$$

Implementing the Weyl and Majorana conditions

$$\tilde{F}^{11}\Psi = \Psi, \quad \Psi^\dagger \tilde{F}^0 = \Psi^T C_{10} \equiv \bar{\Psi}, \quad (\text{A.22})$$

one obtains

$$\begin{aligned} \Psi &= \frac{1}{2} \begin{pmatrix} i \\ 0 \\ 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \chi_1 + i\chi_2 \\ \chi_3 + i\chi_4 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -i \\ 0 \\ 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ \chi_1 - i\chi_2 \\ \chi_3 - i\chi_4 \end{pmatrix} \\ &+ \frac{1}{2} \begin{pmatrix} 0 \\ -i \\ 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ \chi_5 + i\chi_6 \\ \chi_7 + i\chi_8 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ i \\ 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \chi_5 - i\chi_6 \\ \chi_7 - i\chi_8 \\ 0 \\ 0 \end{pmatrix}, \end{aligned} \quad (\text{A.23})$$

where the  $\chi_A$  are real 2-spinors. Redefining the fields in terms of 8  $d = 3$  Majorana 2-spinors  $\lambda_A$

$$\begin{aligned} \chi_1 &= \begin{pmatrix} (\lambda_1)_1 \\ (\lambda_5)_1 \end{pmatrix}, \quad \chi_5 = \begin{pmatrix} (\lambda_1)_2 \\ (\lambda_5)_2 \end{pmatrix}, \quad \chi_3 = \begin{pmatrix} (\lambda_3)_1 \\ (\lambda_7)_1 \end{pmatrix}, \quad \chi_7 = \begin{pmatrix} (\lambda_3)_2 \\ (\lambda_7)_2 \end{pmatrix}, \\ \chi_2 &= \begin{pmatrix} (\lambda_2)_1 \\ (\lambda_6)_1 \end{pmatrix}, \quad \chi_6 = -\begin{pmatrix} (\lambda_2)_2 \\ (\lambda_6)_2 \end{pmatrix}, \quad \chi_4 = \begin{pmatrix} (\lambda_4)_1 \\ (\lambda_8)_1 \end{pmatrix}, \quad \chi_8 = -\begin{pmatrix} (\lambda_4)_2 \\ (\lambda_8)_2 \end{pmatrix}, \end{aligned} \quad (\text{A.24})$$

where the index outside the bracket denotes the first or second component of the spinor, one obtains from the  $\mathcal{N} = 1$ ,  $d = 10$  action

$$S_{\mathcal{N}=1, d=10} = \frac{1}{g^2} \text{Tr} \int d^{10}x \left( -\frac{1}{2} F_{\bar{M}\bar{N}} F^{\bar{M}\bar{N}} + i\bar{\Psi} \tilde{F}^{\bar{M}} D_{\bar{M}} \Psi \right), \quad (\text{A.25})$$

the  $\mathcal{N} = 8$ ,  $d = 3$  action

$$S_{\mathcal{N}=8, d=3} = \frac{1}{g^2} \text{Tr} \int d^3x \left( -\frac{1}{2} F_{\bar{M}\bar{N}} F^{\bar{M}\bar{N}} + i\bar{\lambda}_A \gamma^\mu D_\mu \lambda_A + \rho_{AB}^i \bar{\lambda}_A [\Phi_i, \lambda_B] \right), \quad (\text{A.26})$$

where  $\rho_{AB}^C$  are the matrices relating the  $\mathbf{8}_v$  (index  $C = 1, \dots, 8$ ),  $\mathbf{8}_s$  (index  $A = 1, \dots, 8$ ), and  $\mathbf{8}_c$  (index  $B = 1, \dots, 8$ ) of  $SO(8)$ . Explicitly we have that  $A_{\bar{M}} = (A_\mu, \Phi_i)$ ,  $i = 2, \dots, 8$  and

$$\begin{aligned} \rho_{AB}^C &= \left\{ \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \mathbb{1} \otimes \mathbb{1} \otimes i\sigma^2, -\sigma^1 \otimes i\sigma^2 \otimes \sigma^3, \sigma^3 \otimes i\sigma^2 \otimes \sigma^3, -i\sigma^2 \otimes \mathbb{1} \otimes \sigma^3, \right. \\ &\quad \left. i\sigma^2 \otimes \sigma^1 \otimes \sigma^1, \mathbb{1} \otimes i\sigma^2 \otimes \sigma^1, -i\sigma^2 \otimes \sigma^3 \otimes \sigma^1 \right\}. \end{aligned} \quad (\text{A.27})$$



One has that  $\rho_{AC}^D \rho_{BC}^E + \rho_{AC}^E \rho_{BC}^D = 2\delta^{DE} \delta_{AB}$ ,  $\rho_{AB}^i = -\rho_{BA}^i$ .

The supersymmetry of the theory may be gotten by following steps similar to those in section A.3.1. The supercharge (B.10) may be used (taking the  $\mathcal{N} = 1$  theory in  $d = 10$ ) along with the decomposition given in (A.23) and (A.24). The results may be compactly expressed as

$$Q_A^\alpha |a_B\rangle = \frac{1}{2} u^\alpha \rho_{AC}^B |\lambda_C\rangle, \quad Q_A^\alpha |\lambda_B\rangle = -\frac{1}{2} u^\alpha \rho_{AB}^C |a_C\rangle. \quad (\text{A.28})$$

## B Supercharge for the $\mathcal{N} = 1$ theory

We begin with the Lagrangian

$$\mathcal{L} = -\frac{1}{2} F_{MN} F^{MN} + i\bar{\Psi} \Gamma^M D_M \Psi. \quad (\text{B.1})$$

The SUSY variations of the fields are as follows

$$\delta A_N = -2i\bar{\Psi} \Gamma_N \epsilon, \quad \delta \Psi = F_{PQ} \Gamma^{PQ} \epsilon. \quad (\text{B.2})$$

The variation of the action is then (knowing that the interacting theory is supersymmetric, we set the coupling to zero and so take  $D_M \rightarrow \nabla_M$  (to remain as general as possible we use the  $\nabla$  in place of the partial derivative))

$$\delta \mathcal{L} = 4iF^{MN} \nabla_M (\bar{\Psi} \Gamma_N \epsilon) + i\bar{\Psi} \Gamma^M \nabla_M (F_{PQ} \Gamma^{PQ} \epsilon) + i \overline{(F_{PQ} \Gamma^{PQ} \epsilon)} \Gamma^M \nabla_M \Psi. \quad (\text{B.3})$$

The bar operation is  $\bar{\Psi} \equiv \Psi^T C$ , where  $C^T = -C$  and  $C \Gamma^M C^{-1} = -(\Gamma^M)^T$ . Using integration by parts on the second term in (B.3) and then reversing the order in the last term one finds

$$\delta \mathcal{L} = 4iF^{MN} \nabla_M (\bar{\Psi} \Gamma_N \epsilon) - 2i (\nabla_M \bar{\Psi}) \Gamma^M F_{PQ} \Gamma^{PQ} \epsilon + \nabla_M (i\bar{\Psi} \Gamma^M \Gamma^{PQ} F_{PQ} \epsilon). \quad (\text{B.4})$$

Now we use the fact that

$$\Gamma^M \Gamma^{PQ} F_{PQ} = 2F^{MQ} \Gamma_Q + \Gamma^{MPQ} F_{PQ}, \quad (\text{B.5})$$

to produce

$$\delta \mathcal{L} = 4iF^{MN} \bar{\Psi} \Gamma_N \nabla_M \epsilon - 2i (\nabla_M \bar{\Psi}) \Gamma^{MPQ} F_{PQ} \epsilon + \nabla_M (i\bar{\Psi} \Gamma^M \Gamma^{PQ} F_{PQ} \epsilon). \quad (\text{B.6})$$

Now notice that  $\Gamma^{MPQ} \nabla_M F_{PQ}$  is identically zero since we have antisymmetrized double partial derivatives acting on the gauge field in the field strength. Thus, integration by parts on the middle term above produces

$$\begin{aligned} \delta \mathcal{L} &= 4iF^{MN} \bar{\Psi} \Gamma_N \nabla_M \epsilon + 2i\bar{\Psi} \Gamma^{MPQ} F_{PQ} \nabla_M \epsilon \\ &\quad + \nabla_M (-2i\bar{\Psi} \Gamma^{MPQ} F_{PQ} \epsilon) + \nabla_M (i\bar{\Psi} \Gamma^M \Gamma^{PQ} F_{PQ} \epsilon) \\ &= 2i\bar{\Psi} \Gamma^M \Gamma^{PQ} F_{PQ} \nabla_M \epsilon + \nabla_M (i\bar{\Psi} (\Gamma^M \Gamma^{PQ} - 2\Gamma^{MPQ}) F_{PQ} \epsilon), \end{aligned} \quad (\text{B.7})$$

where we have made use of (B.5) in the second equality. In flat space  $\nabla_M \epsilon = 0$  and we can build the conserved Noether current associated with the symmetry as usual

$$j^M = \frac{\delta \mathcal{L}}{\delta (\partial_M A_N)} \delta A_N + \frac{\delta \mathcal{L}}{\delta (\partial_M \Psi)} \delta \Psi - \mathcal{J}^M, \quad (\text{B.8})$$

where  $\mathcal{J}^M$  is the total derivative arising from the variation of the action, i.e.  $\delta\mathcal{L} = \nabla_M \mathcal{J}^M$ . We therefore find

$$\begin{aligned} j^M &= 4iF^{MN}\bar{\Psi}\Gamma_N\epsilon + i\bar{\Psi}\Gamma^M\Gamma^{PQ}\epsilon F_{PQ} - (i\bar{\Psi}(\Gamma^M\Gamma^{PQ} - 2\Gamma^{MPQ})F_{PQ}\epsilon) \\ &= 4iF^{MN}\bar{\Psi}\Gamma_N\epsilon + 2i\bar{\Psi}\Gamma^{MPQ}F_{PQ}\epsilon \\ &= 2i\bar{\Psi}\Gamma^M\Gamma^{PQ}F_{PQ}\epsilon \end{aligned} \tag{B.9}$$

where we have made use of (B.5) in the last equality. The supercharge is then given by

$$Q = \int_{\text{space}} j^0 = 2i \int_{\text{space}} \bar{\Psi}\Gamma^0\Gamma^{PQ}F_{PQ}\epsilon. \tag{B.10}$$

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